Metric and Hilbert Spaces 15.10.2014 (1)

Topics

(1) Topological spaces, uniform spaces, metric spaces, normed vector spaces, inner product spaces.

(1.5) ~~subspaces~~ Examples of spaces and product spaces and $B(V,W)$ and function spaces.

(2) Functions, Relations, Posets.
Sets, functions, cardinality.

(3) Linear algebra - Vector spaces, bases, Linear transformations
Inner products, eigenvalues and eigenvectors.

(4) ~~Sequences~~ Convergence and series, Hausdorff and compactness

(5) Gram-Schmidt and determinants.

Modules for affine Lie algebras

(1) Category $\mathcal{O}$
(2) Finite dimensional.
(3) Wakimoto modules
(4) Extremal weight modules
(5) Smooth modules
(6) Admissible representations
(7) Weyl modules

Screening operators are intertwiners

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$\{\text{Topological spaces}\}$ M with $\mathcal{T}$

$\cup\!|$

$\{\text{Uniform spaces}\}$ M with $\mathcal{X}$

$\cup\!|$

$\{\text{Metric spaces}\}$ M with d

$\cup\!|$

$\{\text{Normed vector spaces}\}$ M with $\|\cdot\|$

$\cup\!|$

$\{\text{inner product spaces}\}$. M with $\langle\cdot,\cdot\rangle$

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let $\bar{\ }: \mathbb{C} \to \mathbb{C}$ be complex conjugation.
An <u>inner product space</u> is a vector space V over $\mathbb{K}$ with a function

$$\begin{aligned} V \times V &\longrightarrow \mathbb{K} \\ (x,y) &\longmapsto \langle x,y \rangle \end{aligned} \quad \text{such that}$$

(a) If $x,y,z \in V$ then $\langle x+y, z\rangle = \langle x,z\rangle + \langle y,z\rangle$,

(b) If $x,y \in V$ then $\langle x,y\rangle = \overline{\langle y,x\rangle}$,

(c) If $c \in \mathbb{K}$ and $x,y \in V$ then $\langle cx,y\rangle = c\langle x,y\rangle$,

(d) If $x \in V$ then $\langle x,x\rangle \in \mathbb{R}_{\geq 0}$

(e) If $x \in V$ and $\langle x,x\rangle = 0$ then $x = 0$.

<u>Theorem</u> <u>5.1</u> Let V be an inner product space.

(a) If $x,y \in V$ then $|\langle x,y\rangle| \leq \|x\| \cdot \|y\|$

(b) If $x,y \in V$ then $\|x+y\| \leq \|x\| + \|y\|$.

Let V be an inner product space. Define

$$\begin{aligned} V &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto \|x\| \end{aligned} \quad \text{by} \quad \|x\| = \sqrt{\langle x,x\rangle}.$$

Show that V with the function $\|\cdot\|: V \to \mathbb{R}_{\geq 0}$ is a normed vector space.

Proof of (a) Let $x, y \in V$.

Case 1: $y = 0$. Then

$$|\langle x, y \rangle| = 0 \quad \text{and} \quad \|x\| \cdot \|y\| = 0.$$

Case 2: $y \neq 0$. Let ~~$x, y \in V$ and~~

$$a = \langle x, x \rangle, \qquad b = \langle x, y \rangle, \qquad c = \langle y, y \rangle.$$

Let $\lambda \in \mathbb{K}$. Then

$$0 \leq \langle x + \lambda y, x + \lambda y \rangle \qquad \text{(by property (d))}$$

$$= a + b\bar{\lambda} + \bar{b}\lambda + c\lambda\bar{\lambda} \qquad \text{(by (a) and (b) and (c))}$$

Let $\lambda = -\frac{b}{c}$ (using property (e)) so that

$$0 \leq a - \frac{b\bar{b}}{c}$$

Since $c = \langle y, y \rangle > 0$ (because $y \neq 0$), multiplying each side by c gives

$$0 \leq ac - |b|^2 = \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2.$$

So $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$. //

Proof of (b)

By (a):

$$\mathrm{Re}(\langle x,y\rangle) \leq |\langle x,y\rangle| \leq \|x\|\,\|y\|$$

so that

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \|y\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle)$$
$$\leq \|x\|^2 + \|y\|^2 + 2\|x\|\cdot\|y\|$$
$$= (\|x\| + \|y\|)^2.$$

So $\|x+y\| \leq \|x\| + \|y\|$. //.

HW Prove the Cauchy-Schwartz and triangle inequalities for the norms $\| \|_p$ and $\| \|_q$ with $p \in \mathbb{R}_{>0}$ and $\frac{1}{p} + \frac{1}{q} = 1$.

HW Prove the Cauchy-Schwartz and triangle inequalities for the norms $\| \|_\infty$ and $\| \|_1$.

HW What is Lagrange's identity?